

Space-time BRST symmetries

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Abstract Bearing in mind BV quantization of gauge gravitation theory, we extend general covariant transformations to the BRST ones.

1 Introduction

Let us consider a Lagrangian system on a smooth fiber bundle $Y \rightarrow X$. It is called a gauge system if its Lagrangian L admits a set of symmetries depending on parameter functions ξ^r and their derivatives. BRST transformations come from gauge transformations by replacement of gauge parameters with odd ghosts c^r . Moreover, one completes these transformations with the terms acting on ghosts such that the total ones become nilpotent.

In the case of general covariant transformations, parameter functions are vector fields on X . We introduce the corresponding ghosts and construct BRST transformations in the case of metric and metric-affine gravitation theories.

Let $J^r Y$, $r = 1, \dots$, be finite order jet manifolds of sections of $Y \rightarrow X$. In the sequel, the index $r = 0$ stands for Y . Given bundle coordinates (x^λ, y^i) on Y , jet manifolds $J^r Y$ are endowed with the adapted coordinates $(x^\lambda, y^i, y_\Lambda^i)$, where $\Lambda = (\lambda_k \dots \lambda_1)$, $k = 1, \dots, r$, is a symmetric multi-index. We use the notation $\lambda + \Lambda = (\lambda \lambda_k \dots \lambda_1)$ and

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} y_{\lambda+\Lambda}^i \partial_i^\Lambda, \quad d_\Lambda = d_{\lambda_r} \circ \dots \circ d_{\lambda_1}. \quad (1)$$

In order to describe gauge transformations depending on parameters, let us consider Lagrangian formalism on the bundle product

$$E = Y \times_X V, \quad (2)$$

where $V \rightarrow X$ is a vector bundle whose sections are gauge parameter functions [2]. Let $V \rightarrow X$ be coordinated by (x^λ, ξ^r) . Then gauge transformations are represented by a differential operator

$$v = \sum_{0 \leq |\Lambda| \leq m} v_r^{i,\Lambda}(x^\lambda, y_\Sigma^i) \xi_\Lambda^r \partial_i \quad (3)$$

on E (2) which is linear on V and takes its values into the vertical tangent bundle VY of $Y \rightarrow X$. By means of a replacement of even gauge parameters ξ^r and their jets ξ_Λ^r with the odd ghosts c^r and their jets c_Λ^r , the operator (3) defines a graded derivation

$$v = \sum_{0 \leq |\Lambda| \leq m} v_r^{i,\Lambda}(x^\lambda, y_\Sigma^i) c_\Lambda^r \partial_i \quad (4)$$

of the algebra of the original even fields and odd ghosts. Its extension

$$v = \sum_{0 \leq |\Lambda| \leq m} v_r^{i,\Lambda} c_\Lambda^r \partial_i + u^r \partial_r \quad (5)$$

to ghosts is called the BRST transformation if it is nilpotent. Such an extension exists if the original gauge transformations form an algebra [14].

Note that, if gauge transformations are reducible, $(0 \leq k)$ -stage ghosts are introduced [3]. Gauge theories in question here are irreducible.

In the case of gravitation theory, a fiber bundle $Y \rightarrow X$ belongs to the category of natural bundles, and it admits the canonical lift of any vector field on X . One thinks of such a lift as being an infinitesimal generator of general covariant transformations of Y . In this case, the vector bundle $V \rightarrow X$ possesses the composite fibration $V \rightarrow TX \rightarrow X$, where TX is the tangent bundle of X .

Here, we consider the following three gauge theories: (i) the gauge model of principal connections on a principal bundle (the gauge symmetry (32), the BRST symmetry (67)), (ii) this gauge model in the presence of a metric gravity (the gauge symmetry (41), the BRST symmetry (68)) and (iii) in the presence of a metric-affine gravity (the gauge symmetry (46), the BRST symmetry (69)).

2 Gauge systems on fiber bundles

Lagrangian formalism on a fiber bundle $Y \rightarrow X$ is phrased in terms of the following graded differential algebra (henceforth GDA).

With the inverse system of jet manifolds

$$X \xleftarrow{\pi} Y \xleftarrow{\pi_0^1} J^1 Y \xleftarrow{\dots} J^{r-1} Y \xleftarrow{\pi_{r-1}^r} J^r Y \xleftarrow{\dots}, \quad (6)$$

one has the direct system

$$\mathcal{O}^* X \xrightarrow{\pi^*} \mathcal{O}^* Y \xrightarrow{\pi_0^{1*}} \mathcal{O}_1^* Y \longrightarrow \dots \mathcal{O}_{r-1}^* Y \xrightarrow{\pi_{r-1}^{r*}} \mathcal{O}_r^* Y \longrightarrow \dots \quad (7)$$

of GDAs $\mathcal{O}_r^* Y$ of exterior forms on jet manifolds $J^r Y$ with respect to the pull-back monomorphisms π_{r-1}^{r*} . Its direct limit $\mathcal{O}_\infty^* Y$ is a GDA consisting of all exterior forms on finite order jet manifolds modulo the pull-back identification.

The projective limit $(J^\infty Y, \pi_r^\infty : J^\infty Y \rightarrow J^r Y)$ of the inverse system (6) is a Fréchet manifold. A bundle atlas $\{(U_Y; x^\lambda, y^i)\}$ of $Y \rightarrow X$ yields the coordinate atlas

$$\{((\pi_0^\infty)^{-1}(U_Y); x^\lambda, y_\Lambda^i)\}, \quad y_{\lambda+\Lambda}^i = \frac{\partial x^\mu}{\partial x^\lambda} d_\mu y_\Lambda^i, \quad 0 \leq |\Lambda|, \quad (8)$$

of $J^\infty Y$, where d_μ are the total derivatives (1). Then $\mathcal{O}_\infty^* Y$ can be written in a coordinate form where the horizontal one-forms $\{dx^\lambda\}$ and the contact one-forms $\{\theta_\Lambda^i = dy_\Lambda^i - y_{\lambda+\Lambda}^i dx^\lambda\}$ are local generating elements of the $\mathcal{O}_\infty^0 Y$ -algebra $\mathcal{O}_\infty^* Y$. There is the canonical decomposition $\mathcal{O}_\infty^* Y = \oplus \mathcal{O}_\infty^{k,m} Y$ of $\mathcal{O}_\infty^* Y$ into $\mathcal{O}_\infty^0 Y$ -modules $\mathcal{O}_\infty^{k,m} Y$ of k -contact and m -horizontal forms together with the corresponding projectors $h_k : \mathcal{O}_\infty^* Y \rightarrow \mathcal{O}_\infty^{k,*} Y$ and $h^m : \mathcal{O}_\infty^* Y \rightarrow \mathcal{O}_\infty^{*,m} Y$. Accordingly, the exterior differential on $\mathcal{O}_\infty^* Y$ is split into the sum $d = d_H + d_V$ of the nilpotent total and vertical differentials

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad d_V(\phi) = \theta_\Lambda^i \wedge \partial_i^\Lambda \phi, \quad \phi \in \mathcal{O}_\infty^* Y.$$

Any finite order Lagrangian

$$L = \mathcal{L}\omega : J^r Y \rightarrow \bigwedge^n T^* X, \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad n = \dim X, \quad (9)$$

is an element of $\mathcal{O}_\infty^{0,n} Y$, while

$$\delta L = \mathcal{E}_i \theta^i \wedge \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda (\partial_i^\Lambda \mathcal{L}) \theta^i \wedge \omega \in \mathcal{O}_\infty^{1,n} Y \quad (10)$$

is its Euler–Lagrange operator taking values into the vector bundle

$$T^* Y \bigwedge_Y^n T^* X = V^* Y \bigotimes_Y^n T^* X. \quad (11)$$

A Lagrangian system on a fiber bundle $Y \rightarrow X$ is said to be a gauge theory if its Lagrangian L admits a family of variational symmetries parameterized by elements of a vector bundle $V \rightarrow X$ and its jet manifolds as follows.

Let $\mathfrak{d}\mathcal{O}_\infty^0 Y$ be the $\mathcal{O}_\infty^0 Y$ -module of derivations of the \mathbb{R} -ring $\mathcal{O}_\infty^0 Y$. Any $\vartheta \in \mathfrak{d}\mathcal{O}_\infty^0 Y$ yields the graded derivation (the interior product) $\vartheta \rfloor \phi$ of the GDA $\mathcal{O}_\infty^* Y$ given by the relations

$$\vartheta \rfloor df = \vartheta(f), \quad \vartheta \rfloor (\phi \wedge \sigma) = (\vartheta \rfloor \phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (\vartheta \rfloor \sigma), \quad f \in \mathcal{O}_\infty^0 Y, \quad \phi, \sigma \in \mathcal{O}_\infty^* Y,$$

and its derivation (the Lie derivative)

$$\mathbf{L}_\vartheta \phi = \vartheta \rfloor d\phi + d(\vartheta \rfloor \phi), \quad \mathbf{L}_\vartheta (\phi \wedge \phi') = \mathbf{L}_\vartheta (\phi) \wedge \phi' + \phi \wedge \mathbf{L}_\vartheta (\phi'), \quad \phi, \phi' \in \mathcal{O}_\infty^* Y. \quad (12)$$

Relative to an atlas (8), a derivation $\vartheta \in \mathfrak{d}\mathcal{O}_\infty^0$ reads

$$\vartheta = \vartheta^\lambda \partial_\lambda + \vartheta^i \partial_i + \sum_{|\Lambda| > 0} \vartheta_\Lambda^i \partial_i^\Lambda, \quad (13)$$

where the tuple of derivations $\{\partial_\lambda, \partial_i^\Lambda\}$ is defined as the dual of that of the exterior forms $\{dx^\lambda, dy_\Lambda^i\}$ with respect to the interior product \rfloor [11]. Note that the tuple of derivations $\{\partial_i^\Lambda\}$ is the dual of the basis $\{\theta_\Lambda^i\}$ of contact forms.

A derivation ϑ is called contact if the Lie derivative \mathbf{L}_ϑ (12) preserves the contact ideal of the GDA $\mathcal{O}_\infty^* Y$ generated by contact forms. A derivation v (13) is contact iff

$$\vartheta_\Lambda^i = d_\Lambda(\vartheta^i - y_\mu^i \vartheta^\mu) + y_{\mu+\Lambda}^i \vartheta^\mu, \quad 0 < |\Lambda|. \quad (14)$$

Any contact derivation admits the horizontal splitting

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda d_\lambda + (v^i \partial_i + \sum_{0 < |\Lambda|} d_\Lambda v^i \partial_i^\Lambda), \quad v^i = \vartheta^i - y_\mu^i \vartheta^\mu. \quad (15)$$

Its vertical part ϑ_V is completely determined by the first summand

$$v = v^i(x^\lambda, y_\Lambda^i) \partial_i, \quad 0 \leq |\Lambda| \leq k. \quad (16)$$

This is a section of the pull-back $VY \times_Y J^k Y \rightarrow J^k Y$, i.e., a k -order VY -valued differential operator on Y . One calls v (16) a generalized vector field on Y . Any vertical contact derivation ϑ satisfies the relations

$$\vartheta \rfloor d_H \phi = -d_H(\vartheta \rfloor \phi), \quad \mathbf{L}_\vartheta(d_H \phi) = d_H(\mathbf{L}_\vartheta \phi), \quad \phi \in \mathcal{O}_\infty^* Y. \quad (17)$$

One can show that the Lie derivative of a Lagrangian L (9) along a contact derivation ϑ (15) fulfills the first variational formula

$$\mathbf{L}_\vartheta L = v \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + \mathcal{L} d_V(\vartheta_H \rfloor \omega), \quad (18)$$

where Ξ_L is a Lepagean equivalent of L [11]. A contact derivation ϑ (15) is called variational if the Lie derivative (18) is d_H -exact, i.e., $\mathbf{L}_\vartheta L = d_H \sigma$, $\sigma \in \mathcal{O}_\infty^{0,n-1}$. A glance at the expression (18) shows that: (i) ϑ (15) is variational only if it is projected onto X ; (ii) ϑ is variational iff its vertical part ϑ_V is well; (iii) it is variational iff $v \rfloor \delta L$ is d_H -exact. Therefore, we can restrict our consideration to vertical contact derivations $\vartheta = \vartheta_V$. A generalized vector field v (16) is called a variational symmetry of a Lagrangian L if it generates a variational contact derivation.

Turn now to the notion of a gauge symmetry [2]. Let us consider the bundle product E (2) coordinated by (x^λ, y^i, ξ^r) . Given a Lagrangian L on Y , let us consider its pull-back, say again L , onto E . Let ϑ_E be a contact derivation of the \mathbb{R} -ring $\mathcal{O}_\infty^0 E$, whose restriction

$$\vartheta = \vartheta_E|_{\mathcal{O}_\infty^0 Y} = \sum_{0 \leq |\Lambda|} d_\Lambda v^i \partial_i^\Lambda \quad (19)$$

to $\mathcal{O}_\infty^0 Y \subset \mathcal{O}_\infty^0 E$ is linear in coordinates ξ_Ξ^r . It is determined by a generalized vector field v_E on E whose projection

$$v : J^k E \xrightarrow{v_E} VE \rightarrow E \times_Y VY$$

is a linear VY -valued differential operator v (3) on E . Let ϑ_E be a variational symmetry of a Lagrangian L on E , i.e.,

$$v_E \rfloor \delta L = v \rfloor \delta L = d_H \sigma. \quad (20)$$

Then one says that v (3) is a gauge symmetry of a Lagrangian L .

In accordance with Noether's second theorem [2], if a Lagrangian L (9) admits a gauge symmetry v (3), its Euler–Lagrange operator (10) obeys the Noether identity

$$\left[\sum_{0 \leq |\Lambda| \leq m} \Delta_r^{i,\Lambda} d_\Lambda \mathcal{E}_i \right] \xi^r \omega = 0, \quad (21)$$

where

$$\Delta_r^{i,\Lambda} = \sum_{0 \leq |\Sigma| \leq m-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C_{|\Sigma+\Lambda|}^{|\Sigma|} d_\Sigma v_r^{i,\Sigma+\Lambda}. \quad (22)$$

For instance, if a gauge symmetry

$$v = (v_r^i \xi^r + v_r^{i,\mu} \xi_\mu^r) \partial_i \quad (23)$$

is of first jet order in parameters, the corresponding Noether identity (21) reads

$$\Delta_r^i = v_r^i - d_\mu v_r^{i,\mu}, \quad \Delta_r^{i,\mu} = -v_r^{i,\mu}, \quad (24)$$

$$[v_r^i \mathcal{E}_i - d_\mu (v_r^{i,\mu} \mathcal{E}_i)] \xi^r \omega = 0. \quad (25)$$

Gauge models considered below are of this type.

3 Space-time gauge symmetries

We consider gravitation theory in the absence of spinor fields. Its gauge symmetries are general covariant transformations (e.g., [5, 6, 9]). As was mentioned above, gravitation theory is formulated on natural bundles which admit the canonical lift of any vector field on X . One thinks of such a lift as being an infinitesimal generator of general covariant transformations. Natural bundles are exemplified by tensor bundles, the bundles of world metrics and world connections. One also considers non-vertical automorphisms of a principal bundle $P \rightarrow X$ and their extensions to the gauge-natural prolongations of P and the associated natural-gauge bundles [7, 8, 12].

We here address the gauge theory of principal connections on a principal bundle $P \rightarrow X$ with a structure Lie group G . These connections are represented by sections of the quotient

$$C = J^1 P / G \rightarrow X, \quad (26)$$

called the bundle of principal connections [9]. This is an affine bundle coordinated by (x^λ, a_λ^r) such that, given a section A of $C \rightarrow X$, its components $A_\lambda^r = a_\lambda^r \circ A$ are coefficients of the familiar local connection form (i.e., gauge potentials). We consider the GDA $\mathcal{O}_\infty^* C$.

Infinitesimal generators of one-parameter groups of automorphisms of a principal bundle P are G -invariant projectable vector fields on $P \rightarrow X$. They are associated to sections of the vector bundle $T_G P = TP/G \rightarrow X$. This bundle is endowed with the coordinates $(x^\lambda, \tau^\lambda = \dot{x}^\lambda, \xi^r)$ with respect to the fiber bases $\{\partial_\lambda, e_r\}$ for $T_G P$, where $\{e_r\}$ is the basis for the right Lie algebra \mathfrak{g} of G such that $[e_p, e_q] = c_{pq}^r e_r$. If

$$u = u^\lambda \partial_\lambda + u^r e_r, \quad v = v^\lambda \partial_\lambda + v^r e_r, \quad (27)$$

are sections of $T_G P \rightarrow X$, their bracket reads

$$[u, v] = (u^\mu \partial_\mu v^\lambda - v^\mu \partial_\mu u^\lambda) \partial_\lambda + (u^\lambda \partial_\lambda v^r - v^\lambda \partial_\lambda u^r + c_{pq}^r u^p v^q) e_r. \quad (28)$$

Any section u of the vector bundle $T_G P \rightarrow X$ yields the vector field

$$u_C = u^\lambda \partial_\lambda + (c_{pq}^r a_\lambda^p u^q + \partial_\lambda u^r - a_\mu^r \partial_\lambda u^\mu) \partial_r^\lambda \quad (29)$$

on the bundle of principal connections C (26) [9].

Let us consider a subbundle $V_G P = VP/G \rightarrow X$ of the vector bundle $T_G X$ coordinated by (x^λ, ξ^r) . Its sections $u = u^r e_r$ are infinitesimal generators of vertical automorphisms of P . There is the exact sequence of vector bundles

$$0 \rightarrow V_G P \longrightarrow T_G P \rightarrow TX \rightarrow 0.$$

Its pull-back onto C admits the canonical splitting which takes the coordinate form

$$\tau^\lambda \partial_\lambda + \xi^r e_r = \tau^\lambda (\partial_\lambda + a_\lambda^r e_r) + (\xi^r - \tau^\lambda a_\lambda^r) e_r. \quad (30)$$

Let us consider the bundle product

$$E = C \times_X T_G P, \quad (31)$$

coordinated by $(x^\lambda, a_\lambda^r, \tau^\lambda, \xi^r)$. It can be provided with the generalized vector field

$$v_E = v = (c_{pq}^r a_\lambda^p \xi^q + \xi_\lambda^r - a_\mu^r \tau_\lambda^\mu - \tau^\mu a_{\mu\lambda}^r) \partial_r^\lambda, \quad (32)$$

taking the form

$$v = (c_{pq}^r a_\lambda^p \xi'^q + \xi_\lambda'^r + \tau^\mu \mathcal{F}_{\lambda\mu}^r) \partial_r^\lambda, \quad \xi'^r = \xi^r - \tau^\lambda a_\lambda^r, \quad (33)$$

due to the splitting (30). For instance, this is a gauge symmetry of the global Chern–Simons Lagrangian in gauge theory on a principal bundle with a structure semi-simple Lie group

G over a three-dimensional base X [10]. Given a section B of $C \rightarrow X$ (i.e., a background gauge potential), this Lagrangian reads

$$L = \left[\frac{1}{2} a_{mn}^G \varepsilon^{\alpha\beta\gamma} a_\alpha^m (\mathcal{F}_{\beta\gamma}^n - \frac{1}{3} c_{pq}^n a_\beta^p a_\gamma^q) - \frac{1}{2} a_{mn}^G \varepsilon^{\alpha\beta\gamma} B_\alpha^m (F(B)_{\beta\gamma}^n - \frac{1}{3} c_{pq}^n B_\beta^p B_\gamma^q) - d_\alpha (a_{mn}^G \varepsilon^{\alpha\beta\gamma} a_\beta^m B_\gamma^n) \right] d^3x, \quad (34)$$

$$F(B)_{\lambda\mu}^r = \partial_\lambda B_\mu^r - \partial_\mu B_\lambda^r + c_{pq}^r B_\lambda^p B_\mu^q, \quad \mathcal{F}_{\lambda\mu}^r = a_{\lambda\mu}^r - a_{\mu\lambda}^r + c_{pq}^r a_\lambda^p a_\mu^q,$$

where a^G is the Killing form. Its first term is the well-known local Chern–Simons Lagrangian, the second one is a density on X , and the Lie derivative of the third term is d_H -exact because of the relations (17). The corresponding Noether identities (25) read

$$c_{pq}^r a_\lambda^p \mathcal{E}_r^\lambda - d_\lambda (\mathcal{E}_q^\lambda) = 0, \quad (35)$$

$$-a_{\mu\lambda}^r \mathcal{E}_r^\lambda + d_\lambda (a_\mu^r \mathcal{E}_r^\lambda) = 0. \quad (36)$$

The first one is the well-known Noether identity corresponding to the vertical gauge symmetry

$$v = (c_{pq}^r a_\lambda^p \xi^q + \xi_\lambda^r) \partial_r^\lambda. \quad (37)$$

The second Noether identity (36) is brought into the form

$$-a_\mu^r [c_{pq}^r a_\lambda^p \mathcal{E}_r^\lambda - d_\lambda (\mathcal{E}_q^\lambda)] + \mathcal{F}_{\lambda\mu}^r \mathcal{E}_r^\lambda = 0,$$

i.e., it is equivalent to the Noether identity

$$\mathcal{F}_{\lambda\mu}^r \mathcal{E}_r^\lambda = 0, \quad (38)$$

which also comes from the splitting (30) of the generalized vector field v .

In the case of Chern–Simons Lagrangian, the Noether identity (38) is trivial since its coefficients $\mathcal{F}_{\lambda\mu}^r$ vanish on the kernel of the Euler–Lagrange operator

$$\delta L = a_{rn}^G \varepsilon^{\lambda\mu\nu} \mathcal{F}_{\mu\nu}^n \theta_\lambda^r \wedge \omega$$

of the Chern–Simons Lagrangian (34).

In order to obtain a gauge symmetry of the Yang–Mills Lagrangian, one should complete the generalized vector field (32) with the term acting on a world metric.

Let LX the fiber bundle of linear frames in the tangent bundle TX of X . It is a principal bundle with the structure group $GL(n, \mathbb{R})$, $n = \dim X$, which is reduced to its maximal compact subgroup $O(n)$. Global sections of the quotient bundle $\Sigma = LX/O(n)$ are Riemannian metrics on X . If X obeys the well-known topological conditions, pseudo-Riemannian metrics on X are similarly described. Being an open subbundle of the tensor

bundle $\overset{2}{V}TX$, the bundle Σ is provided with bundle coordinates $\sigma^{\mu\nu}$. It admits the canonical lift

$$u_\Sigma = u^\lambda \partial_\lambda + (\sigma^{\nu\beta} \partial_\nu u^\alpha + \sigma^{\alpha\nu} \partial_\nu u^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} \quad (39)$$

of any vector field $u = u^\lambda \partial_\lambda$ on X .

In order to describe the gauge theory of principal connections in the presence of a dynamic metric field, let us consider the bundle product

$$E = C \underset{X}{\times} \Sigma \underset{X}{\times} T_G P, \quad (40)$$

coordinated by $(x^\lambda, a_\lambda^r, \sigma^{\alpha\beta}, \tau^\lambda, \xi^r)$. It can be provided with the generalized vector field

$$v = (c_{pq}^r a_\lambda^p \xi^q + \xi_\lambda^r - a_\mu^r \tau_\lambda^\mu - \tau^\mu a_{\mu\lambda}^r) \partial_r^\lambda + (\sigma^{\nu\beta} \tau_\nu^\alpha + \sigma^{\alpha\nu} \tau_\nu^\beta - \tau^\lambda \sigma_\lambda^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}}. \quad (41)$$

This is a gauge symmetry of the sum $L = L_{\text{YM}} + L_g$ of the Yang–Mills Lagrangian $L_{\text{YM}}(\mathcal{F}_{\alpha\beta}^r, \sigma^{\mu\nu})$ and a Lagrangian L_g of a metric field. The corresponding Noether identities read

$$c_{pq}^r a_\lambda^p \mathcal{E}_r^\lambda - d_\lambda(\mathcal{E}_q^\lambda) = 0, \quad (42)$$

$$-a_{\mu\lambda}^r \mathcal{E}_r^\lambda + d_\lambda(a_\mu^r \mathcal{E}_r^\lambda) - \sigma_\mu^{\alpha\beta} \mathcal{E}_{\alpha\beta} - 2d_\nu(\sigma^{\nu\beta} \mathcal{E}_{\mu\beta}) = 0. \quad (43)$$

The first one is the Noether identity (35), while the second identity is brought into the form

$$-a_{\mu\lambda}^r \mathcal{E}_r^\lambda + d_\lambda(a_\mu^r \mathcal{E}_r^\lambda) - 2\nabla_\nu(\sigma^{\nu\beta} \mathcal{E}_{\mu\beta}) = 0,$$

where ∇_ν are covariant derivatives with respect to the Levi–Civita connection

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu), \quad K_\lambda^\mu{}_\nu = -\frac{1}{2} \sigma^{\nu\beta} (\sigma_{\lambda\beta\mu} + \sigma_{\mu\beta\lambda} - \sigma_{\beta\lambda\mu}).$$

In metric-affine gravitation theory, dynamic variables are a world metric and a world connection. World connections are principal connections on the frame bundle LX , and they are represented by sections of the quotient fiber bundle

$$C_K = J^1 LX / GL(n, \mathbb{R}). \quad (44)$$

This fiber bundle is provided with bundle coordinates $(x^\lambda, k_\lambda^\nu{}_\alpha)$ such that, for any section K of $C_K \rightarrow X$, its coordinates $k_\lambda^\nu{}_\alpha \circ K = K_\lambda^\nu{}_\alpha$ are coefficient of the linear connection

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu)$$

on TX . The bundle of world connections (44) admits the canonical lift

$$u_K = u^\mu \partial_\mu + [\partial_\nu u^\alpha k_\mu^\nu{}_\beta - \partial_\beta u^\nu k_\mu^\alpha{}_\nu - \partial_\mu u^\nu k_\nu^\alpha{}_\beta + \partial_{\mu\beta} u^\alpha] \frac{\partial}{\partial k_\mu^\alpha{}_\beta}.$$

In order to describe the gauge theory of principal connections in the presence of metric-affine gravity, let us consider the bundle product

$$E = C \times_X \Sigma \times_X C_K \times_X T_G P, \quad (45)$$

coordinated by $(x^\lambda, a_\lambda^r, \sigma^{\alpha\beta}, k_\lambda^\nu, \tau^\lambda, \xi^r,)$. It can be provided with the generalized vector field

$$v = (c_{pq}^r a_\lambda^p \xi^q + \xi_\lambda^r - a_\mu^r \tau_\lambda^\mu - \tau^\mu a_{\mu\lambda}^r) \partial_r^\lambda + (\sigma^{\nu\beta} \partial_\nu \tau^\alpha + \sigma^{\alpha\nu} \partial_\nu \tau^\beta - \tau^\lambda \sigma_\lambda^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} \quad (46)$$

$$(\tau_\nu^\alpha k_\mu^\nu{}_\beta - \tau_\beta^\nu k_\mu^\alpha{}_\nu - \tau_\mu^\nu k_\nu^\alpha{}_\beta + \tau_{\mu\beta}^\alpha - \tau^\lambda k_{\lambda\mu}^\alpha{}_\beta) \frac{\partial}{\partial k_\mu^\alpha{}_\beta}.$$

This is a gauge symmetry of the sum $L = L_{\text{YM}} + L_{\text{MA}}$ of the Yang–Mills Lagrangian $L_{\text{YM}}(\mathcal{F}_{\alpha\beta}^r, \sigma^{\mu\nu})$ and a metric-affine Lagrangian L_{MA} .

4 BRST symmetries

In order to introduce BRST symmetries, let us consider Lagrangian systems of even and odd variables. We describe odd variables and their jets on a smooth manifold X as generating elements of the structure ring of a graded manifold whose body is X [11, 13]. This definition reproduces the heuristic notion of jets of ghosts in the field-antifield BRST theory [1, 4].

Recall that any graded manifold (\mathfrak{A}, X) with a body X is isomorphic to the one whose structure sheaf \mathfrak{A}_Q is formed by germs of sections of the exterior product

$$\wedge Q^* = \mathbb{R} \oplus_X Q^* \oplus_X \wedge^2 Q^* \oplus_X \cdots, \quad (47)$$

where Q^* is the dual of some real vector bundle $Q \rightarrow X$ of fiber dimension m . In field models, a vector bundle Q is usually given from the beginning. Therefore, we consider graded manifolds (X, \mathfrak{A}_Q) where the above mentioned isomorphism holds, and call (X, \mathfrak{A}_Q) the simple graded manifold constructed from Q . The structure ring \mathcal{A}_Q of sections of \mathfrak{A}_Q consists of sections of the exterior bundle (47) called graded functions. Let $\{c^a\}$ be the fiber basis for $Q^* \rightarrow X$, together with transition functions $c'^a = \rho_b^a c^b$. It is called the local basis for the graded manifold (X, \mathfrak{A}_Q) . With respect to this basis, graded functions read

$$f = \sum_{k=0}^m \frac{1}{k!} f_{a_1 \dots a_k} c^{a_1} \cdots c^{a_k},$$

where $f_{a_1 \dots a_k}$ are local smooth real functions on X .

Given a graded manifold (X, \mathfrak{A}_Q) , let $\mathfrak{d}\mathcal{A}_Q$ be the \mathcal{A}_Q -module of \mathbb{Z}_2 -graded derivations of the \mathbb{Z}_2 -graded ring of \mathcal{A}_Q , i.e.,

$$u(ff') = u(f)f' + (-1)^{[u][f]} fu(f'), \quad u \in \mathfrak{d}\mathcal{A}_Q, \quad f, f' \in \mathcal{A}_Q,$$

where $[\cdot]$ denotes the Grassmann parity. Its elements are called \mathbb{Z}_2 -graded (or, simply, graded) vector fields on (X, \mathfrak{A}_Q) . Due to the canonical splitting $VQ = Q \times Q$, the vertical tangent bundle $VQ \rightarrow Q$ of $Q \rightarrow X$ can be provided with the fiber bases $\{\partial_a\}$ which is the dual of $\{c^a\}$. Then a graded vector field takes the local form $u = u^\lambda \partial_\lambda + u^a \partial_a$, where u^λ, u^a are local graded functions. It acts on \mathcal{A}_Q by the rule

$$u(f_{a\dots b}c^a \cdots c^b) = u^\lambda \partial_\lambda(f_{a\dots b})c^a \cdots c^b + u^d f_{a\dots b} \partial_d](c^a \cdots c^b). \quad (48)$$

This rule implies the corresponding transformation law

$$u'^\lambda = u^\lambda, \quad u'^a = \rho_j^a u^j + u^\lambda \partial_\lambda(\rho_j^a) c^j.$$

Then one can show that graded vector fields on a simple graded manifold can be represented by sections of the vector bundle $\mathcal{V}_Q \rightarrow X$, locally isomorphic to $\wedge Q^* \otimes_X (Q \oplus_X TX)$.

Accordingly, graded exterior forms on the graded manifold (X, \mathfrak{A}_Q) are introduced as sections of the exterior bundle $\wedge \mathcal{V}_Q^*$, where $\mathcal{V}_Q^* \rightarrow X$ is the $\wedge Q^*$ -dual of \mathcal{V}_Q . Relative to the dual local bases $\{dx^\lambda\}$ for T^*X and $\{dc^b\}$ for Q^* , graded one-forms read

$$\phi = \phi_\lambda dx^\lambda + \phi_a dc^a, \quad \phi'_a = \rho^{-1b}_a \phi_b, \quad \phi'_\lambda = \phi_\lambda + \rho^{-1b}_a \partial_\lambda(\rho_j^a) \phi_b c^j.$$

The duality morphism is given by the interior product

$$u]\phi = u^\lambda \phi_\lambda + (-1)^{[\phi_a]} u^a \phi_a.$$

Graded exterior forms constitute the bigraded differential algebra (henceforth BGDA) \mathcal{C}_Q^* with respect to the bigraded exterior product \wedge and the exterior differential d .

Since the jet bundle $J^r Q \rightarrow X$ of a vector bundle $Q \rightarrow X$ is a vector bundle, let us consider the simple graded manifold $(X, \mathfrak{A}_{J^r Q})$ constructed from $J^r Q \rightarrow X$. Its local basis is $\{x^\lambda, c_\Lambda^a\}$, $0 \leq |\Lambda| \leq r$, together with the transition functions

$$c'_{\lambda+\Lambda}^a = d_\lambda(\rho_j^a c_\Lambda^j), \quad d_\lambda = \partial_\lambda + \sum_{|\Lambda| < r} c_{\lambda+\Lambda}^a \partial_a^a, \quad (49)$$

where ∂_a^a are the duals of c_Λ^a . Let $\mathcal{C}_{J^r Q}^*$ be the BGDA of graded exterior forms on the graded manifold $(X, \mathfrak{A}_{J^r Q})$. A linear bundle morphism $\pi_{r-1}^r : J^r Q \rightarrow J^{r-1} Q$ yields the corresponding monomorphism of BGDA's $\mathcal{C}_{J^{r-1} Q}^* \rightarrow \mathcal{C}_{J^r Q}^*$. Hence, there is the direct system of BGDA's

$$\mathcal{C}_Q^* \xrightarrow{\pi_0^{1*}} \mathcal{C}_{J^1 Q}^* \cdots \xrightarrow{\pi_{r-1}^{r*}} \mathcal{C}_{J^r Q}^* \longrightarrow \cdots. \quad (50)$$

Its direct limit $\mathcal{C}_\infty^* Q$ consists of graded exterior forms on graded manifolds $(X, \mathfrak{A}_{J^r Q})$, $r \in \mathbb{N}$, modulo the pull-back identification, and it inherits the BGDA operations intertwined by the monomorphisms π_{r-1}^{r*} . It is a $C^\infty(X)$ -algebra locally generated by the elements $(1, c_\Lambda^a, dx^\lambda, \theta_\Lambda^a = dc_\Lambda^a - c_{\lambda+\Lambda}^a dx^\lambda)$, $0 \leq |\Lambda|$.

In order to regard even and odd dynamic variables on the same footing, let $Y \rightarrow X$ be hereafter an affine bundle, and let $\mathcal{P}_\infty^* Y \subset \mathcal{O}_\infty^* Y$ be the $C^\infty(X)$ -subalgebra of exterior forms whose coefficients are polynomial in the fiber coordinates y_Λ^i on jet bundles $J^r Y \rightarrow X$. Let us consider the product

$$\mathcal{S}_\infty^* = \mathcal{C}_\infty^* Q \wedge \mathcal{P}_\infty^* Y \quad (51)$$

of graded algebras $\mathcal{C}_\infty^* Q$ and $\mathcal{P}_\infty^* Y$ over their common graded subalgebra $\mathcal{O}^* X$ of exterior forms on X [11]. It consists of the elements

$$\sum_i \psi_i \otimes \phi_i, \quad \sum_i \phi_i \otimes \psi_i, \quad \psi \in \mathcal{C}_\infty^* Q, \quad \phi \in \mathcal{P}_\infty^* Y,$$

modulo the commutation relations

$$\begin{aligned} \psi \otimes \phi &= (-1)^{|\psi||\phi|} \phi \otimes \psi, & \psi \in \mathcal{C}_\infty^* Q, & \quad \phi \in \mathcal{P}_\infty^* Y, \\ (\psi \wedge \sigma) \otimes \phi &= \psi \otimes (\sigma \wedge \phi), & \sigma \in \mathcal{O}^* X. \end{aligned} \quad (52)$$

They are endowed with the total form degree $|\psi| + |\phi|$ and the total Grassmann parity $[\psi]$. Their multiplication

$$(\psi \otimes \phi) \wedge (\psi' \otimes \phi') := (-1)^{|\psi'||\phi|} (\psi \wedge \psi') \otimes (\phi \wedge \phi'). \quad (53)$$

obeys the relation

$$\varphi \wedge \varphi' = (-1)^{|\varphi||\varphi'| + [\varphi][\varphi']} \varphi' \wedge \varphi, \quad \varphi, \varphi' \in \mathcal{S}_\infty^*,$$

and makes \mathcal{S}_∞^* (51) into a bigraded $C^\infty(X)$ -algebra. For instance, elements of the ring \mathcal{S}_∞^0 are polynomials of c_Λ^a and y_Λ^i with coefficients in $C^\infty(X)$.

The algebra \mathcal{S}_∞^* is provided with the exterior differential

$$d(\psi \otimes \phi) := (d_C \psi) \otimes \phi + (-1)^{|\psi|} \psi \otimes (d_P \phi), \quad \psi \in \mathcal{C}_\infty^*, \quad \phi \in \mathcal{P}_\infty^*, \quad (54)$$

where d_C and d_P are exterior differentials on the differential algebras $\mathcal{C}_\infty^* Q$ and $\mathcal{P}_\infty^* Y$, respectively. It obeys the relations

$$d(\varphi \wedge \varphi') = d\varphi \wedge \varphi' + (-1)^{|\varphi|} \varphi \wedge d\varphi', \quad \varphi, \varphi' \in \mathcal{S}_\infty^*,$$

and makes \mathcal{S}_∞^* into a BGDA. Hereafter, let the collective symbols s_Λ^A stand both for even and odd generating elements c_Λ^a , y_Λ^i of the $C^\infty(X)$ -ring \mathcal{S}_∞^0 . Then the BGDA \mathcal{S}_∞^* is locally generated by $(1, s_\Lambda^A, dx^\lambda, \theta_\Lambda^A = ds_\Lambda^A - s_{\lambda+\Lambda}^A dx^\lambda)$, $|\Lambda| \geq 0$. We agree to call elements of \mathcal{S}_∞^* the graded exterior forms on X .

Similarly to $\mathcal{O}_\infty^* Y$, the BGDA \mathcal{S}_∞^* is decomposed into \mathcal{S}_∞^0 -modules $\mathcal{S}_\infty^{k,r}$ of k -contact and r -horizontal graded forms together with the corresponding projections h_k and h^r . Accordingly, the exterior differential d (54) on \mathcal{S}_∞^* is split into the sum $d = d_H + d_V$ of the total and vertical differentials

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad d_V(\phi) = \theta_\Lambda^A \wedge \partial_A^\Lambda \phi, \quad \phi \in \mathcal{S}_\infty^*.$$

One can think of the elements

$$L = \mathcal{L}\omega \in \mathcal{S}_\infty^{0,n}, \quad \delta(L) = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^A \wedge d_\Lambda(\partial_A^\Lambda L) \in \mathcal{S}_\infty^{0,n}$$

as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

A graded derivation $\vartheta \in \mathfrak{d}\mathcal{S}_\infty^0$ of the \mathbb{R} -ring \mathcal{S}_∞^0 is said to be contact if the Lie derivative \mathbf{L}_ϑ preserves the ideal of contact graded forms of the BGDA \mathcal{S}_∞^* . With respect to the local basis $(x^\lambda, s_\Lambda^A, dx^\lambda, \theta_\Lambda^A)$ for the BGDA \mathcal{S}_∞^* , any contact graded derivation takes the form

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda d_\lambda + (\vartheta^A \partial_A + \sum_{|\Lambda| > 0} d_\Lambda \vartheta^A \partial_A^\Lambda), \quad (55)$$

where $\vartheta^\lambda, \vartheta^A$ are local graded functions [11]. The interior product $\vartheta \rfloor \phi$ and the Lie derivative $\mathbf{L}_\vartheta \phi$, $\phi \in \mathcal{S}_\infty^*$, are defined by the formulae

$$\begin{aligned} \vartheta \rfloor \phi &= \vartheta^\lambda \phi_\lambda + (-1)^{[\phi]_A} \vartheta^A \phi_A, & \phi &\in \mathcal{S}_\infty^1, \\ \vartheta \rfloor (\phi \wedge \sigma) &= (\vartheta \rfloor \phi) \wedge \sigma + (-1)^{|\phi| + [\phi][\vartheta]} \phi \wedge (\vartheta \rfloor \sigma), & \phi, \sigma &\in \mathcal{S}_\infty^*, \\ \mathbf{L}_\vartheta \phi &= \vartheta \rfloor d\phi + d(\vartheta \rfloor \phi), & \mathbf{L}_\vartheta(\phi \wedge \sigma) &= \mathbf{L}_\vartheta(\phi) \wedge \sigma + (-1)^{[\vartheta][\phi]} \phi \wedge \mathbf{L}_\vartheta(\sigma). \end{aligned}$$

The Lie derivative $\mathbf{L}_\vartheta L$ of a Lagrangian L along a contact graded derivation ϑ (55) fulfills the first variational formula

$$\mathbf{L}_\vartheta L = \vartheta_V \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + d_V(\vartheta_H \rfloor \omega) \mathcal{L}, \quad (56)$$

where $\Xi_L = \Xi + L$ is a Lepagean equivalent of a graded Lagrangian L [11].

A contact graded derivation ϑ is said to be variational if the Lie derivative (56) is d_H -exact. A glance at the expression (56) shows that: (i) a contact graded derivation ϑ is variational only if it is projected onto X , and (ii) ϑ is variational iff its vertical part ϑ_V is well. Therefore, we restrict our consideration to vertical contact graded derivations

$$\vartheta = \sum_{0 \leq |\Lambda|} d_\Lambda v^A \partial_A^\Lambda. \quad (57)$$

Such a derivation is completely defined by its first summand

$$v = v^A(x^\lambda, s_\Lambda^A) \partial_A, \quad 0 \leq |\Lambda| \leq k, \quad (58)$$

which is also a graded derivation of \mathcal{S}_∞^0 . It is called the generalized graded vector field. A glance at the first variational formula (56) shows that ϑ (57) is variational iff $v \lrcorner \delta L$ is d_H -exact.

A vertical contact graded derivation ϑ (57) is said to be nilpotent if

$$\mathbf{L}_v(\mathbf{L}_v\phi) = \sum_{|\Sigma| \geq 0, |\Lambda| \geq 0} (v_\Sigma^B \partial_B^\Sigma (v_\Lambda^A) \partial_A^\Lambda + (-1)^{[s^B][v^A]} v_\Sigma^B v_\Lambda^A \partial_B^\Sigma \partial_A^\Lambda) \phi = 0 \quad (59)$$

for any horizontal graded form $\phi \in \mathcal{S}_\infty^{0,*}$ or, equivalently, $(\vartheta \circ \vartheta)(f) = 0$ for any graded function $f \in \mathcal{S}_\infty^0$. One can show that ϑ is nilpotent only if it is odd and iff the equality

$$\vartheta(v^A) = \sum_{|\Sigma| \geq 0} v_\Sigma^B \partial_B^\Sigma (v^A) = 0 \quad (60)$$

holds for all v^A [11].

Return now to the original gauge system on a fiber bundle Y with a Lagrangian L (9) and a gauge symmetry v (3). For the sake of simplicity, $Y \rightarrow X$ is assumed to be affine. Let us consider the BGDA $\mathcal{S}_\infty^*[V; Y] = \mathcal{C}_\infty^* V \wedge \mathcal{P}_\infty^* Y$ locally generated by $(1, c_\Lambda^r, dx^\lambda, y_\Lambda^i, \theta_\Lambda^r, \theta_\Lambda^i)$. Let $L \in \mathcal{O}_\infty^{0,n} Y$ be a polynomial in y_Λ^i , $0 \leq |L|$. Then it is a graded Lagrangian $L \in \mathcal{P}_\infty^{0,n} Y \subset \mathcal{S}_\infty^{0,n}[V; Y]$ in $\mathcal{S}_\infty^*[V; Y]$. Its gauge symmetry v (3) gives rise to the generalized vector field $v_E = v$ on E , and the latter defines the generalized graded vector field v (58) by the formula (4). It is easily justified that the contact graded derivation ϑ (57) generated by v (4) is variational for L . It is odd, but need not be nilpotent. However, one can try to find a nilpotent contact graded derivation (57) generated by some generalized graded vector field (5) which coincides with ϑ on $\mathcal{P}_\infty^* Y$. Then v (5) is called a BRST symmetry.

In this case, the nilpotency conditions (60) read

$$\sum_\Sigma d_\Sigma \left(\sum_{\Xi} v_r^{i,\Xi} c_{\Xi}^r \right) \sum_\Lambda \partial_i^\Sigma (v_s^{j,\Lambda}) c_\Lambda^s + \sum_\Lambda d_\Lambda (u^r) v_r^{j,\Lambda} = 0, \quad (61)$$

$$\sum_\Lambda \left(\sum_{\Xi} d_\Lambda (v_r^{i,\Xi} c_{\Xi}^r) \partial_i^\Lambda + d_\Lambda (u^r) \partial_r^\Lambda \right) u^q = 0 \quad (62)$$

for all indices j and q . They are equations for graded functions $u^r \in \mathcal{S}_\infty^0[V; Y]$. Since these functions are polynomials

$$u^r = u_{(0)}^r + \sum_\Gamma u_{(1)p}^{r,\Gamma} c_\Gamma^p + \sum_{\Gamma_1, \Gamma_2} u_{(2)p_1 p_2}^{r, \Gamma_1 \Gamma_2} c_{\Gamma_1}^{p_1} c_{\Gamma_2}^{p_2} + \dots \quad (63)$$

in c_Λ^s , the equations (61) – (62) take the form

$$\sum_\Sigma d_\Sigma \left(\sum_{\Xi} v_r^{i,\Xi} c_{\Xi}^r \right) \sum_\Lambda \partial_i^\Sigma (v_s^{j,\Lambda}) c_\Lambda^s + \sum_\Lambda d_\Lambda (u_{(2)}^r) v_r^{j,\Lambda} = 0, \quad (64)$$

$$\sum_\Lambda d_\Lambda (u_{(k \neq 2)}^r) v_r^{j,\Lambda} = 0, \quad (65)$$

$$\sum_\Lambda \sum_{\Xi} d_\Lambda (v_r^{i,\Xi} c_{\Xi}^r) \partial_i^\Lambda u_{(k-1)}^q + \sum_{m+n-1=k} d_\Lambda (u_{(m)}^r) \partial_r^\Lambda u_{(n)}^q = 0. \quad (66)$$

One can think of the equalities (64) and (66) as being the generalized commutation relations and generalized Jacobi identities of original gauge transformations, respectively [14].

5 Space-time BRST symmetries

Let us consider gauge symmetries (32), (41) and (46). Following the procedure in Section 4, we replace parameters ξ^r and τ^λ with the odd ghosts c^r and c^λ , respectively, and obtain the generalized graded vector fields

$$v = (c_{pq}^r a_\lambda^p c^q + c_\lambda^r - a_\mu^r c_\lambda^\mu - c^\mu a_{\mu\lambda}^r) \partial_r^\lambda + (-\frac{1}{2} c_{pq}^r c^p c^q - c^\mu c_\mu^r) \partial_r + c_\mu^\lambda c^\mu \partial_\lambda, \quad (67)$$

$$v = (c_{pq}^r a_\lambda^p c^q + c_\lambda^r - a_\mu^r c_\lambda^\mu - c^\mu a_{\mu\lambda}^r) \partial_r^\lambda + (\sigma^{\nu\beta} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta - c^\lambda \sigma_\lambda^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (-\frac{1}{2} c_{pq}^r c^p c^q - c^\mu c_\mu^r) \partial_r + c_\mu^\lambda c^\mu \partial_\lambda, \quad (68)$$

$$v = (c_{pq}^r a_\lambda^p c^q + c_\lambda^r - a_\mu^r c_\lambda^\mu - c^\mu a_{\mu\lambda}^r) \partial_r^\lambda + (\sigma^{\nu\beta} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta - c^\lambda \sigma_\lambda^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (c_\nu^\alpha k_\mu{}^\nu{}_\beta - c_\beta^\nu k_\mu{}^\alpha{}_\nu - c_\mu^\nu k_\nu{}^\alpha{}_\beta + c_{\mu\beta}^\alpha - c^\lambda k_{\lambda\mu}{}^\alpha{}_\beta) \frac{\partial}{\partial k_\mu{}^\alpha{}_\beta} + (-\frac{1}{2} c_{pq}^r c^p c^q - c^\mu c_\mu^r) \partial_r + c_\mu^\lambda c^\mu \partial_\lambda. \quad (69)$$

The vertical contact graded derivations (57) generated by these generalized graded vector fields are nilpotent, i.e., these generalized graded vector fields are BRST symmetries.

It should be noted that all the BRST symmetries (67) – (69) possesses the same ghost term

$$(-\frac{1}{2} c_{pq}^r c^p c^q - c^\mu c_\mu^r) \partial_r + c_\mu^\lambda c^\mu \partial_\lambda.$$

It is not surprised because this term corresponds to the bracket of the vector fields (27).

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